Modalité Renforcée 16

Brice HANNEBICQUE, Gansheng TAN & Yuguang ZHOU

march 19$^{\text{th}}$, Tuesday

Exercise 1. Let X, Y be two iid real-valued random variables whose distribution has a density f with respect to the Lebesgue measure.

1. (a) Prove that almost surely, $X \neq 0$.
   (b) Prove that one can define the random variable $Z = Y/X$.
2. Show that the distribution of Z has a density with respect to the Lebesgue measure and give its expression.

---

1. (a)
$$\mathbf{P}(X=0) = \mathbf{E}\left(\mathbb{1}_{\{0\}}(X)\right) = \int_{\mathbb{R}} \mathbb{1}_{\{0\}}(x) f(x)\lambda(\mathrm{d}x) = 0.$$

   (b) Let us denote $\Omega^* = \{X \neq 0\}$ and define
$$\forall \omega \in \Omega,\ Z(\omega) = \begin{cases} Y(\omega)/X(\omega) & \text{if } \omega \in \Omega^*, \\ 0 & \text{otherwise.} \end{cases}$$

   Since $\mathbf{P}(\Omega^*)$, Z is equal to Y/X almost surely.
   Moreover, Z is a random variable since it can be written as
$$Z = \Phi(X,Y)\mathbb{1}_{\Omega^*}$$
   where $\Phi : (\mathbb{R}^*)^2 \ni (x,y) \mapsto y/x$ is a continuous, hence measurable, map.

2. Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded and measurable map, then
$$\begin{aligned}
\mathbf{E}(h(Z)) &= \int_{\mathbb{R}^2} h(y/x) f_{(X,Y)}(x,y)\ \lambda(\mathrm{d}x,\mathrm{d}y) && \text{by transfert theorem,} \\
&= \int_{\mathbb{R}^2\setminus\{0\}} h(y/x) f(x) f(y)\ \lambda(\mathrm{d}x,\mathrm{d}y) && \text{since } X, Y \overset{\perp}{\sim} f, \\
&= \int_{\mathbb{R}^2\setminus\{0\}} h(u) f(v) f(uv)|v|\ \lambda(\mathrm{d}u,\mathrm{d}v) && \text{by the change of variables,} \\
&= \int_{\mathbb{R}} h(u) \left[\int_{\mathbb{R}} f(uv) f(v)|v|\ \lambda(\mathrm{d}v)\right] \lambda(\mathrm{d}u) && \text{by Fubini's theorem.}
\end{aligned}$$

   Hence, Z admits $\rho(u) = \int_{\mathbb{R}} f(uv)f(v)|v|\ \lambda(\mathrm{d}v)$ has a density. Let us justify the change of variables by considering
$$\begin{array}{cccc}
\Phi : & \mathbb{R}^2\setminus\{0\} & \longrightarrow & \mathbb{R}^2\setminus\{0\} \\
& (x,y) & \longmapsto & (y/x, x) \\
& (v, uv) & \longleftarrow & (u,v).
\end{array}$$

   Obviously, Φ and Φ^{-1} are $\mathcal{C}^1$ functions, hence Φ is a $\mathcal{C}^1$-diffeomorphism. The jacobian is obtained by the straightforward computation
$$|J_{\Phi^{-1}}|(u,v) = \begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} = |v|.$$

Exercise 2. Let $N, X_1, X_2, \dots$ be independent random variables such that $N \sim \mathcal{P}(\lambda)$ and $X_1, X_2, \dots$ have the same distribution μ. Let us define
$$Z = \sum_{k=1}^{N} X_k.$$

Compute the characteristic function ϕ_Z of Z.

The fact that Z is a random variable will be treated in TD. Let $\xi \in \mathbb{R}$, then

$$\phi_Z(\xi) = \mathbf{E}\left(e^{i\xi Z}\right) = \mathbf{E}\left(e^{i\xi Z}\sum_{n=0}^{\infty} \mathbb{1}_{N=n}\right).$$

By Fubini's theorem, one gets

$$\phi_Z(\xi) = \sum_{n=0}^{\infty} \mathbf{E}\left(e^{i\xi Z}\mathbb{1}_{N=n}\right).$$

Since $N, X_1, ..., X_n$ are independent random variables, one may simplify each term as

$$\mathbf{E}\left(e^{i\xi Z}\mathbb{1}_{N=n}\right) = \mathbf{E}\left(e^{i\xi\sum_{k=1}^{n} X_k}\mathbb{1}_{N=n}\right) = \mathbf{E}\left(\prod_{k=1}^{n} e^{i\xi X_k}\right)\mathbf{E}\left(\mathbb{1}_{N=n}\right) = \left(\prod_{k=1}^{n}\mathbf{E}\left(e^{i\xi X_k}\right)\right)\mathbf{P}\left(N=n\right) = \widehat{\mu}(\xi)^n \times e^{-\lambda}\frac{\lambda^n}{n!}$$

where $\widehat{\mu}(\xi) = \int_{\mathbb{R}} e^{i\xi x}\mu(\mathrm{d}x)$ is the characteristic function of X_1 (and hence, any X_k since they have the same distribution μ). Hence, one gets

$$\phi_Z(\xi) = e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda\widehat{\mu}(\xi))^n}{n!} = \exp\left[\lambda(\widehat{\mu}(\xi)-1)\right].$$

Exercise 3. Let U, V be two iid random variables uniformly distributed over $[0,1]$. For all $\delta > 0$, define

$$X_\delta = \left((1+\delta V)\cos(2\pi U), (1+\delta V)\sin(2\pi U)\right).$$

1. (a) Justify why X_δ is random vector.
   (b) Compute the distribution of X_δ and show that it admits a density.
2. (a) Prove that X_δ converges almost surely to a random variable X as $\delta \to 0$.
   (b) Show that the distribution of X does not have a density with respect to the Lebesgue measure.

---

1. (a) X_δ is the composition of the random vector (U, V) with a continuous map, hence X_δ is measurable.
   (b) Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a bounded and measurable map. Then, by a linear change of variables and a polar change of variables, one gets

$$\begin{aligned}
\mathbf{E}\left(h(X_\delta)\right) &= \int_{(0,1)^2} h\left((1+\delta v)\cos(2\pi u), (1+\delta v)\sin(2\pi u)\right)\lambda(\mathrm{d}u, \mathrm{d}v) \\
&= \frac{1}{2\pi\delta}\int_{(1,1+\delta)\times(0,2\pi)} h(r\cos\theta, r\sin\theta)\lambda(\mathrm{d}r, \mathrm{d}\theta) \\
&= \frac{1}{2\pi\delta}\int_{A_\delta} h(x,y)\frac{1}{\sqrt{x^2+y^2}}\lambda(\mathrm{d}x, \mathrm{d}y).
\end{aligned}$$

   Hence, X_δ admits $f_\delta(x,y) = \frac{1}{2\pi\delta\sqrt{x^2+y^2}}\mathbb{1}_{A_\delta}(x,y)$ as a density where $A_\delta = \left\{(x,y) : 1 \leqslant x^2+y^2 \leqslant (1+\delta)^2\right\}$.
2. (a) Let $\omega \in \Omega$, then

$$X_\delta(\omega) = \left((1+\delta V(\omega))\cos(2\pi U(\omega)), (1+\delta V(\omega))\sin(2\pi U(\omega))\right) \underset{\delta\to 0}{\longrightarrow} \left(\cos(2\pi U(\omega)), \sin(2\pi U(\omega))\right) = X(\omega).$$

   Hence X_δ converges to X almost surely as $\delta \to 0$.
   (b) Due to the expression of X, one has

$$\mathbf{P}\left(X \in \mathbf{S}^1\right) = 1.$$

   If X admits a density f, then, since $\lambda(\mathbf{S}^1) = 0$, one would have

$$\mathbf{P}\left(X \in \mathbf{S}^1\right) = \int_{\mathbf{S}^1} f(x)\lambda(\mathrm{d}x) = 0,$$

   which is absurd.